1.5 Differentiability

For more than one variables, there are two types of derivatives: directional and partial.

1.5.1 Successive Differentiation

▶ Let $y = x^5$. So, $\frac{dy}{dx} = 5x^4$, $\frac{d^2y}{dx^2} = 20x^3$, $\frac{d^3y}{dx^3} = 60x^2$, $\frac{d^4y}{dx^4} = 120x$, $\frac{d^5y}{dx^5} = 120$, $\frac{d^6y}{dx^6} = 0$.
▶ Let $y = x^n \Rightarrow \frac{d^ny}{dx^n} = n!$.
▶ Let $y = e^{ax} \Rightarrow \frac{d^ny}{dx^n} = a^n e^{ax}$.
▶ Let $y = \frac{1}{x+a} \Rightarrow \frac{d^ny}{dx^n} = \frac{(-1)^n n!}{(x+a)^{n+1}}$.
▶ Let $y = \sin ax \Rightarrow \frac{d^ny}{dx^n} = a^n \sin(\frac{n\pi}{2} + ax)$.
▶ Let $y = a^x \Rightarrow \frac{d^ny}{dx^n} = a^x(\ln a)^n$.
▶ Let $y = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \Rightarrow \frac{d^ny}{dx^n} = \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$.

Theorem 1.1. *Leibnitz's Theorem:*
$$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + \binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \binom{n}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \cdots + \binom{n}{n-1}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + u\frac{d^nv}{dx^n}$$
or, $$\frac{d^n(uv)}{dx^n} = u_nv + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + \binom{n}{n-1}uv_{n-1} + uv_n$$

[Do It Yourself] 1.11. *If $y = xe^{ax}$ then show that $y_n = a^{n-1}e^{ax}(ax+n)$.*

[Do It Yourself] 1.12. *Find $y^{(n)}$ for i) $y = (ax+b)^m$, $m > n$; ii) $y = \ln(ax+b)$; iii) $y = \sin(ax+b)$; iv) $y = \cos(ax+b)$; v) $y = \sin^2 x$; vi) $y = \sin 2x \cos 4x$; vii) $y = \frac{1}{x^2-3x+2}$; viii) $y = \frac{x^2}{x^2-3x+2}$; ix) $y = e^{ax}\sin(bx+c)$.*
[Ans : *i)* $\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$; *iv)* $a^n\cos(ax+b+\frac{n\pi}{2})$; *ix)* $e^{ax}(a^2+b^2)^{n/2}\sin(bx+c+n\tan^{-1}\frac{b}{a})$]

1.5.2 Directional and Partial Derivative

■ $D_\alpha f(a,b) = \lim_{\rho\to 0}\frac{f(a+\rho\cos(\alpha), b+\rho\sin(\alpha)) - f(a,b)}{\rho}$, is called the directional derivative of $f(x,y)$ at (a,b) in the direction α.
▷ If $\alpha = 0$, $D_\alpha f(a,b) = \lim_{\rho\to 0}\frac{f(a+\rho,b) - f(a,b)}{\rho} = f_x(a,b)$.
▷ If $\alpha = \frac{\pi}{2}$, $D_\alpha f(a,b) = \lim_{\rho\to 0}\frac{f(a,b+\rho) - f(a,b)}{\rho} = f_y(a,b)$.

■ Partial derivative of $f(x,y)$ w.r.t x is defined as: $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$.

▷ Partial derivative of $f(x,y)$ with respect to x at (a,b) is $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$.

▷ Partial derivative of $f(x,y)$ w.r.t y is defined as: $f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$.

▷ Partial derivative of $f(x,y)$ with respect to y at (a,b) is $f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k}$.

Example 1.7. *Show that*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

has partial derivative at $(0,0)$ but not directional derivative in any arbitrary direction.

$\Rightarrow$ *Here* $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$

Also $f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = \lim_{h\to 0} \frac{0}{k} = 0.$

□ *Let us take any arbitrary direction making an angle α with $x-axis$.*

Now, $D_\alpha f(0,0) = \lim_{\rho\to 0} \frac{f(0+\rho\cos(\alpha), 0+\rho\sin(\alpha)) - f(0,0)}{\rho} = \lim_{\rho\to 0} \frac{\sin\alpha\cos\alpha}{\rho}$.

Therefore, $D_\alpha f(0,0)$ *does not exist.*

Example 1.8. *Show that*

$$f(x,y) = \begin{cases} \frac{x^3y}{x^6+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

has directional derivative in any arbitrary direction at $(0,0)$.

$\Rightarrow$ *Let us take any arbitrary direction making an angle α with $x-axis$.*

Now, $D_\alpha f(0,0) = \lim_{\rho\to 0} \frac{f(0+\rho\cos(\alpha), 0+\rho\sin(\alpha)) - f(0,0)}{\rho} = \frac{\cos^3\alpha}{\sin^2\alpha}$.

$\underline{If\ \alpha = 0}$, $D_\alpha f(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$

Therefore, $D_\alpha f(0,0)$ *exist in every direction.*

[Do It Yourself] 1.13. *If*

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & \text{if } x \neq 0,\ y \neq 0 \\ x^2 \sin\frac{1}{x} & \text{if } x \neq 0,\ y = 0 \\ y^2 \sin\frac{1}{y} & \text{if } x = 0,\ y \neq 0 \\ 0 & \text{if } x = 0,\ y = 0 \end{cases}$$

then find $f_x(0,y),\ f_y(x,0)$.

$\left[Hint:\ f_x(0,y) = \lim_{x\to 0} \dfrac{f(x,y) - f(0,y)}{x} = \lim_{x\to 0} x \sin\dfrac{1}{x} = 0\right]$.

Example 1.9. *If*

$$f(x,y) = \begin{cases} x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

$\Rightarrow\ f_{xy}(0,0) = \lim_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h}$.

Now $f_y(h,0) = \lim_{k\to 0} \dfrac{f(h,k) - f(h,0)}{k} = h^2 \lim_{k\to 0} \dfrac{\tan^{-1}\frac{k}{h}}{k} - 0 = h^2 \lim_{k\to 0} \dfrac{\frac{1}{1+\frac{k^2}{h^2}}\frac{1}{h}}{1} = h$.

$f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0$. *So* $f_{xy}(0,0) = 1$.

Similarly, we can show that $f_{yx}(0,0) = -1$. *It implies* $f_{xy}(0,0) \neq f_{yx}(0,0)$.

[Do It Yourself] 1.14. *Let*

$$f(x,y) = \begin{cases} \sqrt{x^2+y^2}\,\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then at the point $(0,0)$

(A) f *is continuous and* f_x, f_y *exist.* *(B)* f *is continuous and* f_x, f_y *do not exist.*

(C) f *is not continuous and* f_x, f_y *exist.* *(D)* f *is not continuous and* f_x, f_y *do not exist.*

$[Hint:\ Easy]$

1.5.4 Total Differentiation

- $f(x,y,z)$ is a function of 3 variables $\Rightarrow$ $\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$.
- $f_{xy} = \frac{\partial}{\partial x}(f_y)$.
- $f_{xx}(a,b) = \lim\limits_{h\to 0} \dfrac{f_x(a+h,b) - f_x(a,b)}{h}$.
- $f_{xy}(a,b) = \lim\limits_{h\to 0} \dfrac{f_y(a+h,b) - f_y(a,b)}{h}$.

1.6 Application of Derivatives

Derivatives can be applied in various fields such as finding maxima-minima, limit, tangent-normal, radius of curvature, graph plotting etc. Here we will study its usage on finding limit of a real-valued function (one variable) and on maxima-minima problem.

1.6.1 Indeterminate Form

Suppose $f(x) = \frac{g(x)}{h(x)}$, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \dfrac{g(x)}{h(x)} = \dfrac{\lim\limits_{x\to a} g(x)}{\lim\limits_{x\to a} h(x)}$, exists if both limit exists and $\lim\limits_{x\to a} h(x) \neq 0$. Now if $\lim\limits_{x\to a} g(x) = 0$ and $\lim\limits_{x\to a} h(x) = 0$, then $\lim\limits_{x\to a} f(x)$ is a $\frac{0}{0}$ indeterminate form. There are various indeterminate forms like $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 0^0, ∞^0, $1^{\pm\infty}$ and the limiting values can be obtain through *L' Hospital's Rule.*

Theorem 1.4. *L' Hospital's Rule ($\frac{0}{0}$): If f, g be two real valued functions such that*

1. *$f^{(n)}, g^{(n)}$ exists in $N'(a,\delta)$ and $g^{(n)} \neq 0$.*
2. *$\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} f'(x) = \cdots = \lim\limits_{x\to a} f^{(n-1)}(x) = 0$ and*
   $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} g'(x) = \cdots = \lim\limits_{x\to a} g^{(n-1)}(x) = 0$.
3. *$\lim\limits_{x\to a} \dfrac{f^{(n)}(x)}{g^{(n)}(x)}$ exists and equal to l.*

Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = l$

Example 1.15. *Find* $\lim_{x\to 0} \dfrac{x-\sin(x)}{\tan^3(x)}$.

$\Rightarrow$ *Here the limit is of the form* $(\frac{0}{0})$, *so we will use L' Hospital Rule.*

$$\lim_{x\to 0} \frac{x-\sin(x)}{\tan^3(x)} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \frac{1-\cos(x)}{3\tan^2(x)\sec^2(x)} \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to 0} \frac{\sin(x)}{6\tan(x)\sec^4(x) + 6\tan^3(x)\sec^2(x)} \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x\to 0} \frac{\cos(x)}{6\sec^6(x) + 42\tan^2(x)\sec^4(x) + 12\tan^4(x)\sec^2(x)} = \frac{1}{6}.$$

[Do It Yourself] 1.18. *Find* $\lim_{x\to 0} \dfrac{e^x + \sin(x) - 1}{\log(1+x)}$, $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2\log(1+x)}{x\sin(x)}$.

Example 1.16. *Find* a, b *such that* $\lim_{x\to 0} \dfrac{a\sin(2x) - b\sin(x)}{x^3} = 1$.

$\Rightarrow$ *Here the limit is of the form* $(\frac{0}{0})$, *so we will use L' Hospital Rule.*

$$\lim_{x\to 0} \frac{a\sin(2x) - b\sin(x)}{x^3} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \frac{2a\cos(2x) - b\cos(x)}{3x^2} \left(\frac{0}{0}\ form\ if\ 2a - b = 0\right)$$

$$= \lim_{x\to 0} \frac{-4a\sin(2x) + b\sin(x)}{6x} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \frac{-8a\cos(2x) + b\cos(x)}{6} = -a.$$

Therefore, $a = -1, b = -2$.

[Do It Yourself] 1.19. *Find* a, b, c *such that* $\lim_{x\to 0} \dfrac{ae^x - b\cos(x) + ce^{-x}}{x\sin(x)} = 2$.

Indeterminate Form $\dfrac{\infty}{\infty}$

Example 1.17. *Find* $\lim_{x\to 0} \log_{\tan^2(x)} \tan^2(2x)$.

$\Rightarrow$ *The given limit is* $\lim_{x\to 0} \log_{\tan^2(x)} \tan^2(2x) = \lim_{x\to 0} \dfrac{\log\tan^2(2x)}{\log\tan^2(x)} = \lim_{x\to 0} \dfrac{\log\tan(2x)}{\log\tan(x)}$.

Here the limit is of the form $(\frac{\infty}{\infty})$, *so we will use L' Hospital Rule.*

$$So,\ \lim_{x\to 0} \frac{\log\tan(2x)}{\log\tan(x)} \left(\frac{\infty}{\infty}\ form\right) = \lim_{x\to 0} \frac{\frac{2\sec^2(2x)}{\tan(2x)}}{\frac{\sec^2(x)}{\tan(x)}} = \lim_{x\to 0} \frac{2\sec^2(2x)\tan(x)}{\sec^2(x)\tan(2x)} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{\sin(2x)\cos(2x)}$$

$$= \lim_{x\to 0} \frac{2\sin(2x)}{\sin(4x)} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \frac{4\cos(2x)}{4\cos(4x)} = 1.$$

[Do It Yourself] 1.20. *Show that* $\lim_{x\to 0} \log_{\cot^2(x)} x^2 = -1$.

$$\boxed{Indeterminate\ Forms:\ \infty-\infty, 0\times\infty, 0^0, \infty^0, 1^{\pm\infty}}$$

Note that, any above form can be reduced to either $\left(\frac{0}{0}\right)$ or, $\left(\frac{\infty}{\infty}\right)$ and then solve.

[Do It Yourself] 1.21. *Find the following limits*

(A) $\lim\limits_{x\to 0} x\log\sin^2(x)$ ♠ $0\times\infty$ *form, reduce* (∞/∞), $\frac{\log\sin^2(x)}{1/x}$ *[Ans : 0].*

(B) $\lim\limits_{x\to\pi/2}(1-\sin(x))\tan(x)$ ♠ *[Ans : 0].*

(C) $\lim\limits_{x\to 0}\left(\frac{1}{\sin^2(x)}-\frac{1}{x^2}\right)$ ♠ $\infty-\infty$ *form, reduce* $(0/0)$, $\frac{x^2-\sin^2(x)}{x^2\sin^2(x)}$ *[Ans : 1/3].*

(D) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\cot(x)\right)$ ♠ *[Ans : 0].*

[Do It Yourself] 1.22. *Find* a, b, c *such that* $\lim\limits_{x\to 0}\frac{a\sin(x)-bx+cx^2+x^3}{2x^2\log(1+x)-2x^3+x^4}$ *is finite. Hence find the limit.*

[Do It Yourself] 1.23. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a differentiable function such that* f' *is continuous on* $\mathbb{R}$ *with* $f'(3)=18$. *Define* $g_n(x)=n[f(x+\frac{5}{n})-f(x-\frac{2}{n})]$. *Then find* $\lim\limits_{n\to\infty} g_n(3)$. $[Hint:\ put\ u=\frac{1}{n}]$

[Do It Yourself] 1.24. *The value of* $\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2}e^{-2n}$ *is*

(A) e^{-2} *(B)* e^{-1} *(C)* e *(D)* e^2

[Hint : $L=\left(1+\frac{2}{n}\right)^{n^2}e^{-2n}\Rightarrow ln(L)=n^2\ln(1+\frac{2}{n})-2n\Rightarrow\lim\limits_{n\to\infty}ln(L)=\lim\limits_{n\to\infty}n^2\ln(1+\frac{2}{n})-2n$

$\Rightarrow\lim\limits_{n\to\infty}ln(L)=\lim\limits_{x\to 0}\frac{1}{x^2}\ln(1+2x)-\frac{2}{x}\Rightarrow ln\left(\lim\limits_{n\to\infty}L\right)=\lim\limits_{x\to 0}\frac{\ln(1+2x)-2x}{x^2}$]

[Do It Yourself] 1.25. *For a suitable* $\alpha > 0$, $\lim_{x\to 0}\left(\frac{1}{e^{2x}-1} - \frac{1}{\alpha x}\right)$ *exists and equal to a finite limit* l. *Then*
(A) $\alpha = 2,\ l = 2.$ *(B)* $\alpha = 2,\ l = -1/2.$ *(C)* $\alpha = 1/2,\ l = -2.$ *(D)* $\alpha = 1/2,\ l = 1/2.$

[Do It Yourself] 1.26. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be differentiable function with* $\lim_{x\to\infty} f(x) = \infty$ *and* $\lim_{x\to\infty} f'(x) = 2$. *Then find the value of* $\lim_{x\to\infty}\left(1 + \frac{f(x)}{x^2}\right)^x$.
$\left[\underline{Hint}: \ \lim_{x\to\infty}\left(1+\frac{f(x)}{x}\right)^x = Exp[\lim_{x\to\infty} f(x)]\right]$

[Do It Yourself] 1.27. *Find* $\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x\tan x}\right]$.

[Do It Yourself] 1.28. *Find* $\lim_{n\to\infty}\left[n - \frac{n}{e}\left(1+\frac{1}{n}\right)^n\right]$.
$[Hint: \ put\ \frac{1}{n} = x]$